On a characterization of an invariant Gaussian measure for linear semigroups

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Preliminaries

$$X_0 \sim \mu \Rightarrow X_t = T_t(X_0) \sim \mu \circ T_t^{-1}.$$

μ is $(T_t)-$invariant if:

$$\mu(T_t^{-1}(E)) = \mu(E), \ \forall t > 0, \ \forall E \in \mathcal{B}(X)$$

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Invariance$\sim$ Generator! Question raised by R. Rudnicki

Theorem

(EL M 2015) A centred Gaussian measure μ, on X with covariance operator R_μ, is invariant if and only if $AR_\mu + R_\mu A^* = 0$ holds on $D(A^*)$.

Proof.

Assume μ invariant, $R_\mu = T(t)R_\mu T^*(t)$, for all $t \geq 0$, and let $x^* \in D(A^*)$.

$$\lim_{t\to 0}\langle \frac{1}{t}(T^*(t) - I)x^*, x\rangle = \langle A^*x^*, x\rangle, \text{ for all } x \in X.$$

By uniform boundednes principle $(\frac{1}{t}(T^*(t) - I)x^*)_{0<t<1}$ is bounded in X^*.

$$(\forall(t_n) \to 0)\ \frac{1}{t_n}(T(t_n)R_\mu x^* - R_\mu x^*) = \frac{1}{t_n}(T(t_n)R_\mu x^* - T(t_n)R_\mu T^*(t_n)x^*),$$
$$= \frac{1}{t_n}T(t_n)R_\mu(x^* - T^*(t_n)x^*).$$

Proof

$R_\mu : \ X^* \to \ X$ compact:
$\exists (t_{n_k}) \ (R_\mu \frac{1}{t_{n_k}}(x^* - T^*(t_{n_k})x^*))_k \to w$.
Claim: $w = -R_\mu A^* x^*$,
For y^* arbitrarily in X^*. Write

$$\begin{aligned} \langle y^*, w\rangle &= \lim_{k\to\infty} \langle y^*, R_\mu \frac{1}{t_{n_k}}(x^* - T^*(t_{n_k})x^*)\rangle, \\ &= \lim_{k\to\infty} \langle \frac{1}{t_{n_k}}(x^* - T^*(t_{n_k})x^*), R_\mu y^*\rangle, \\ &= \langle -A^*x^*, R_\mu y^*\rangle, \\ &= \langle y^*, -R_\mu A^* x^*\rangle. \end{aligned}$$

We deduce that,

$$\forall (t_n) \to 0, \ \exists (t_{n_k}) \ \lim_k \frac{1}{t_{n_k}}(T(t_{n_k})R_\mu x^* - R_\mu x^*) = -R_\mu A^* x^*$$

Finally, $R_\mu x^* \in D(A)$ and $AR_\mu x^* = -R_\mu A^* x^*$.

The Converse's Proof

Let $t > 0$, $h \neq 0$ and $x^* \in D(A^*)$. We have

$$\frac{1}{h}(T_{t+h}R_\mu T^*_{t+h}x^* - T_tR_\mu T^*_tx^*) = T_{t+h}R_\mu\frac{1}{h}[T^*_{t+h} - T^*_t]x^*, \\ + \frac{1}{h}[T_{t+h} - T_t]R_\mu T^*_tx^*.$$

The first term converges to $T_tR_\mu A^*T^*_tx^*$ (Compactness again!.)
The second term converges to $T_tAR_\mu T^*_tx^*$.
we obtain

$$\lim_{h\to 0}\frac{1}{h}(T(t+h)R_\mu T^*(t+h)x^* - T(t)R_\mu T^*(t)x^*) = T(t)[R_\mu A^* + AR_\mu]T^*x^* = 0.$$

Hence for all $t > 0$,

$$T(t)R_\mu T^*(t) = R_\mu, \text{ on, } D(A^*).$$

Converse's Proof

To conclude:
For all $y^* \in X^*$,

$$\langle \cdot, R_\mu y^* \rangle = \langle \cdot, T(t) R_\mu T^*(t) y^* \rangle,$$

on the *weak**-dense subspace $D(A^*)$.
The two mapping are *weak**-continuous
Thus,

$$R_\mu y^* = T(t) R_\mu T^*(t) y^*$$

holds for all $y^* \in X^*$.

Mixing in $L^p(\Omega, d\sigma)$ (joint work with K.Latrach(DIE 2013))

$(\Omega, \mathfrak{B}, \sigma)$ σ-finite measure space, and let $X := L^p(\Omega, d\sigma)$, $1 \leq p < +\infty$. Assume that X is separable, and A is the generator of a C_0-semigroup $T(\cdot)$ on X. Discrete case(F. Bayart and É. Matheron, *Dynamics of linear operators*, Cambridge Tracts in Maths vol 179, (2009).)

Theorem

Assume $\sigma_p(A) \cap i\mathbb{R} \subset i(\omega_1, \omega_2)$ for some ω_1 and ω_2 , and there is a measurable function $u : (\omega_1, \omega_2) \mapsto X$ satisfying the following conditions:

(i) $u_s := u(s) \in \ker(is - A)$ *for a.e.* $s \in (\omega_1, \omega_2)$,

(ii) $(\int_{\omega_1}^{\omega_2} | u_s(\cdot) |^2 \, ds)^{\frac{1}{2}} = v(\cdot) \in L^p(\Omega)$,

(iii) *span*$\{u_s, \ s \in (\omega_1, \omega_2) \setminus N\}$ *is dense in* X *for every subset* N *with zero Lebesgue measure.*

Then there exists an invariant Gaussian measure ν, such that supp$(\nu) = X$ with respect to which $T(\cdot)$ is strong mixing.

Mixing Translation in $L^p(I, \rho(x)dx)$

$$T(t)f(x) = f(x+t),\ x \in I,\ t \geq 0$$

Proposition

If $\int_I \rho(x)dx < \infty$, then there exists an invariant Gaussian measure with full support with respect to which $T(\cdot)$ is strong mixing.

abnormal cell division model

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = -\frac{\partial(xu(t,x))}{\partial x} + \gamma(x)u(t,x) - \beta(x)u(t,x) + 4\beta(2x)\ u(t,2x)\ \chi_{(0,\frac{1}{2})}(x), \\ u(0,\cdot) = \phi \in L^1(0,1). \end{cases}$$

$$v(y) = u(e^{-y},\ y > 0)$$

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Under$\gamma - 3\beta > 0$, $0 \leq \beta \leq \frac{1}{2}$ The solution is chaotic + existence of a strongly mixing non degenerate Gaussian measure.

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Thank you for your attention